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APPLICATION OF NUMERICAL SOLUTIONS TO NON-LINEAR PROBLEMS OF FLUID
MECHANICS OBTAINED BY THE METHOD OF LEAST SQUARES AND THE FINITE
ELEMENT METHOD TO UNSTEADY NAVIER-STOKES EQUATIONS

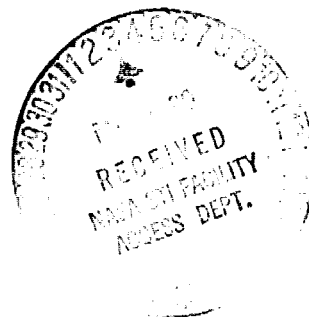
B. Mantel, J. Periaux, P. Perrier

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APPLICATION OF NUMERICAL SOLUTIONS TO NON-LINEAR PROBLEMS OF FLUID MECHANICS OBTAINED BY THE METHOD OF LEAST SQUARES AND THE FINITE ELEMENT METHOD TO UNSTEADY NAVIER-STOKES EQUATIONS

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0. Introduction

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The objective of this report is to present a new method of calculating viscous fluid flows having an average Reynolds number. The actual flows encountered in aeronautics have very large Reynolds numbers, but experience shows that the main flow structures are relatively independent from the Reynolds number.

In the short term, data extracted by a calculation using an average Reynolds number, such as the point of separation, the development and importance of a separated flow, are of interest to the aerodynamics specialist. In the long term, however, the numerical simulation of turbulent separated flows with large Reynolds numbers is searched for when refining an efficient and reliable Reynolds number, by adding to the Navier-Stokes equations either a standard turbulence model (TSEN-SEROU |1|) or a homogeneous random turbulence (PERRIER-PIRONNEAU |2|).

Furthermore, an effective numerical algorithm of 3-D unsteady dimensionless Navier-Stokes equations (1) is of interest for industry.

$$\begin{aligned} \frac{\partial \vec{u}}{\partial t} + \nabla \cdot (\vec{u} \otimes \vec{u}) &= -\vec{\nabla} p + \vec{f} \\ \vec{\nabla} \cdot \vec{u} &= 0 \\ \vec{u}/\Gamma &= \vec{z} \quad ; \quad \vec{u}(x, 0) = \vec{u}_0(x) \end{aligned} \tag{1}$$

* Numbers in the margin indicate pagination in the foreign text.

In (1) \vec{u} designates the flow velocity, p the pressure, \vec{f} the external forces, ν the fluid viscosity ($\nu = \frac{1}{Re}$ Re = Reynolds number), \vec{z} is the velocity specification on the rigid or infinite boundary Γ of the Ω volume occupied by the fluid, \vec{u}_0 the flow velocity at the initial instant $t = 0$.

We propose the solution of (1) by the optimal control theory /2 in terms of a system with distributed parameters (J-L LIONS [3]) and thereby to eliminate the two penalizing defects of the existing traditional methods:

- d1) Setting of parameters to ensure the convergence of the algorithm
- d2) Requirement of a very large computer.

The decoupling method of the equations proposed (series of Stokes problems) is automatic (in the direction of d1) and requires a minimum band width (in the direction of d2). On the other hand, as it is more sophisticated, it shall be more difficult to program.

The complexity of the aerodynamic geometries studied explains the preference given to a quantification of (1) into FINITE ELEMENTS.

The large number of unknowns in industrial applications requires a rapid convergence (cr) for the treatment of non-linearities and schemes implicit in time (cr).

I. QUANTIFICATION IN TIME AND TREATMENT OF NON LINEARITY

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To quantify the time derivative in (1), use is made of a scheme in two steps which is completely implicit and unconditionally stable (CRANK-NICHOLSON). If $\vec{u}^{n+1}(x) = \vec{u}(x, (n+1) \Delta t)$ designates the velocity at instant $(n+1) \Delta t$, with Δt time step, \vec{u}^{n+1} is to be derived from \vec{u}^n , \vec{u}^{n-1} as the solution in (\vec{u}, p) of (2)

$$\begin{aligned} \alpha \vec{u} - \nu \Delta \vec{u} + (\vec{u} \cdot \vec{\nabla}) \vec{u} &= -\vec{\nabla} p + \vec{f}; \quad \vec{\nabla} \cdot \vec{u} = 0 \text{ in } \Omega \\ \vec{u}|_{\Gamma} &= \vec{z} \end{aligned} \quad (2)$$

with $\alpha = \frac{3}{2} \Delta t$ et $\vec{f} = \frac{2\vec{u}^n}{\Delta t} - \frac{\vec{u}^{n-1}}{\Delta t}$

To solve the non-linear system (2), a least squares method in a Sobolev $H^{-1}(\Omega)$, functional space, proposed in (GLOWINSKI-MANTEL-MERIAUX-PIRONNEAU [4]), is given in (3)

$$\begin{aligned} \min_{\vec{u} \in W_2^1(\Omega)} \quad & \| (\alpha - \nu \Delta) \vec{u} + (\vec{u} \cdot \vec{\nabla}) \vec{u} - \vec{f} \|^2_{H^{-1}(\Omega)^N / (\vec{\nabla} p / p \in L^2(\Omega))} \end{aligned} \quad (3)$$

or $H^{-1}(\Omega)^N$ designates the dual space of $H_0^1(\Omega)^N = \{ \vec{v} \in L^2(\Omega)^N / \vec{\nabla} \cdot \vec{v} \in L^2(\Omega)^{N \times N}, \vec{v}|_{\Gamma} = 0 \}$
 $N = 2$ or 3 dimension of the region occupied by the fluid

$$W_2^1(\Omega) = \{ \vec{u} \in H^1(\Omega)^N, \vec{u}|_{\Gamma} = \vec{z} / \vec{\nabla} \cdot \vec{u} = 0 \} \quad (4)$$

It is easy to see that if $H_0^1(\Omega)$ contains a scalar product /4

$$\langle \vec{u}, \vec{v} \rangle = \alpha (\vec{u}, \vec{v}) + \nu (\vec{\nabla} \vec{u}, \vec{\nabla} \vec{v}) \quad (5)$$

where (\cdot, \cdot) designates the scalar product in $L^2(\Omega)$

$$(f, g) = \int_{\Omega} f g \, dx \quad (6)$$

whereas (3) has the structure of an optimal control problem (8)

$$\begin{aligned} \min_{\vec{v} \in W_2^1} J(\vec{v}) &= \frac{\alpha}{2} \int_{\Omega} |\vec{u} - \vec{v}|^2 \, dx + \frac{\nu}{2} \int_{\Omega} |\vec{\nabla}(\vec{u} - \vec{v})|^2 \, dx \\ &= \langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle \end{aligned} \quad (7)$$

in which J is the cost function or criterion and where \vec{u} is a function of \vec{v} via state equation (8)

$$\begin{aligned} \alpha \vec{u} - \nu \Delta \vec{u} &= \vec{v} \pi = \vec{f} - (\vec{v} \cdot \vec{\nabla}) \vec{v} \\ \vec{\nabla} \cdot \vec{u} &= 0 \\ \vec{u}|_{\Gamma} &= \vec{0} \end{aligned} \quad (*) \quad (8)$$

π acting as a pressure of which the variational formulation is (9)

$$\begin{aligned} \alpha \int_{\Omega} \vec{v} \cdot \vec{\eta} \, dx + \nu \int_{\Omega} \vec{\nabla} \vec{v} \cdot \vec{\nabla} \vec{\eta} \, dx &= \int_{\Omega} \vec{f} \cdot \vec{\eta} \, dx - \int_{\Omega} \vec{\eta} \cdot (\vec{v} \cdot \vec{\nabla}) \vec{v} \, dx \\ \vec{v} &\in W_2^1; \quad \vec{\eta} \in W_0^1 \end{aligned} \quad (9)$$

(*) It may be noted that the state system is a modified Stokes problem. (GB4P[5] may be consulted to find the equivalence between the least squares -- optimal control methods).

II. SOLUTION OF AN OPTIMAL CONTROL PROBLEM BY THE CONJUGATE GRADIENT METHOD /5

The solution of (7) and (8) is searched for the the POLAK-RIBIERE [6] version of the conjugate gradient algorithm. It exists is 3 steps:

0. Initialization

Select $\vec{v}^0 \in W_2^1$ (Stokes or idealized fluid)

Calculate \vec{g}^0 as the solution to the variational equation (10)

$$\begin{aligned} \alpha \int_{\Omega} \vec{g}^0 \cdot \vec{\eta} \, dx + \nu \int_{\Omega} \vec{\nabla} \vec{g}^0 \cdot \vec{\nabla} \vec{\eta} \, dx &= \langle J'(\vec{v}^0), \vec{\eta} \rangle \quad \vec{\eta} \in W_0^1 \\ \vec{g}^0 &\in W_0^1 \end{aligned} \quad (10)$$

and where

$$\vec{h}^0 = \vec{g}^0$$

1. Descent (minimization to one variable)

For $m > 0$, assuming $(\vec{v}^m, \vec{g}^m, \vec{h}^m)$ as known, compute

$(\vec{v}^{m+1}, \vec{g}^{m+1}, \vec{h}^{m+1})$ by

$$\lambda^m = \text{Arg min}_{\lambda > 0} J(\vec{v}^m - \lambda \vec{h}^m); \vec{v}^m, \vec{h}^m \text{ set} \quad (11)$$

and where

$$\vec{v}^{m+1} = \vec{v}^m - \lambda^m \vec{h}^m \quad (12)$$

2. Computation of a new descent direction

Define \vec{g}^{m+1} as the solution to the variational equation (13)

$$\alpha \int_{\Omega} \vec{g}^{m+1} \cdot \vec{h} \, dx + \nu \int_{\Omega} \vec{\nabla} \vec{g}^{m+1} \cdot \vec{\nabla} \vec{h} \, dx = \langle J'(\vec{v}^{m+1}), \vec{h} \rangle \quad (13)$$

$$\vec{g}^{m+1} \in W_0; \forall \vec{h} \in W_0$$

Compute γ^{m+1} in (14)

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$$\gamma^{m+1} = \frac{\alpha \int_{\Omega} \vec{g}^{m+1} (\vec{g}^{m+1} - \vec{g}^m) \, dx + \nu \int_{\Omega} \vec{\nabla} \vec{g}^{m+1} \cdot \vec{\nabla} (\vec{g}^{m+1} - \vec{g}^m) \, dx}{\alpha \int_{\Omega} |\vec{g}^m|^2 \, dx + \nu \int_{\Omega} |\vec{\nabla} \vec{g}^m|^2 \, dx}$$

where $\vec{h}^{m+1} = \vec{g}^{m+1} + \gamma^{m+1} \vec{h}^m$

$m = m+1$ and refer to (11)

Remarks: - (10) and (13) are Stokes equations

-Minimization to one variable (11) is achieved with the method of binary searches or with the Fibonacci (POLAK [6]) method.

-The estimate \vec{g}^{m+1} requires detailing $J'(\vec{v}^{m+1})$

Calculation of J' and \vec{g}^{m+1}

$$\begin{aligned} \delta J = \langle J'(\vec{v}), \delta \vec{v} \rangle &= \alpha \int_{\Omega} (\vec{u} - \vec{v}, \delta(\vec{u} - \vec{v})) \, dx \\ &\quad + \nu \int_{\Omega} \vec{\nabla}(\vec{u} - \vec{v}) \cdot \vec{\nabla}(\vec{u} - \vec{v}) \, dx \\ &\quad + o(\delta \vec{u}) + o(\delta \vec{v}) \end{aligned} \quad (16)$$

where δu is the solution of (17) via (8)

$$\alpha \int_{\Omega} \delta \vec{u} \cdot \vec{\eta} dx + \nu \int_{\Omega} \nabla \delta \vec{u} \cdot \nabla \vec{\eta} dx = - \int_{\Omega} \vec{\eta} \cdot (\delta \vec{v} \cdot \vec{\nabla}) \vec{v} dx - \int_{\Omega} \vec{\eta} \cdot (\vec{v} \cdot \vec{\nabla}) \delta \vec{v} dx$$

$$\delta \vec{u} \in W_0; \quad \forall \vec{\eta} \in W_0 \quad (17)$$

Since $(\vec{u} - \vec{v}) \in W_0$ by using (16) (17) we have

$$\langle J'(\vec{v}), \vec{\eta} \rangle = \alpha \int_{\Omega} (\vec{u} - \vec{v}) \cdot \vec{\eta} + \nu \int_{\Omega} \nabla (\vec{u} - \vec{v}) \cdot \nabla \vec{\eta} + \int_{\Omega} (\vec{u} - \vec{v}) \cdot (\vec{v} \cdot \vec{\nabla}) \vec{\eta} \\ + \int_{\Omega} (\vec{u} - \vec{v}) \cdot (\vec{\eta} \cdot \vec{\nabla}) \vec{v}$$

$$\forall \vec{\eta} \in W_0 \quad (18)$$

To calculate \vec{g}^{m+1} from \vec{v}^{m+1} , we must solve (8) 17
 with $\vec{v} = \vec{v}^{m+1}$ to obtain \vec{u}^{m+1} , whereas by using (18)
 we have $\langle J'(\vec{v}^{m+1}), \vec{\eta} \rangle = \alpha \int_{\Omega} (\vec{u}^{m+1} - \vec{v}^{m+1}) \cdot \vec{\eta} dx + \nu \int_{\Omega} \nabla (\vec{u}^{m+1} - \vec{v}^{m+1}) \cdot \nabla \vec{\eta} dx \\ + \int_{\Omega} (\vec{v}^{m+1} - \vec{u}^{m+1}) \cdot (\vec{v}^{m+1} \cdot \vec{\nabla}) \vec{\eta} dx + \int_{\Omega} (\vec{v}^{m+1} - \vec{u}^{m+1}) \cdot (\vec{\eta} \cdot \vec{\nabla}) \vec{v}^{m+1} dx$
 $\forall \vec{\eta} \in W_0 \quad (19)$

and \vec{g}^{m+1} is obtained from (13) (19)

In conclusion, each iteration of the algorithm (10)....(15) requires the solution of several Stokes problems:

- . The Stokes state equation with $\vec{v} = \vec{v}^{m+1}$ to obtain \vec{u}^{m+1}
- . The gradient problem \vec{g}^{m+1} from \vec{u}^{m+1} , \vec{v}^{m+1}
- . The Stokes problems (~ 3) for the evaluation of the cost function J during minimization to one variable (11). Consequently, an EFFICIENT and RAPID Stokes algorithm is a fundamental tool for solving the Navier-Stokes equations via the least squares methods (3).

III. METHOD OF DECOMPOSING THE STOKES ALGORITHM

(GLOWINSKI-PIRONNEAU [7])

We limit ourselves here to the main concepts. For a more detailed development of the method, [8] may be consulted. The essence of the

method consists in selecting the pressure on the boundary Γ in such a way that $\vec{v} \cdot \vec{u} = 0$ is satisfied in Ω .

Assuming the Stokes problem expressed in (20)

$$\alpha \vec{u} - \Delta \vec{u} = \vec{v} p + \vec{f} ; \nabla \cdot \vec{u} = 0 ; \vec{u}|_{\Gamma} = \vec{u} \quad (20)$$

By taking the divergence of (20), we have

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$$-\Delta p = \vec{v} \cdot \vec{f} \quad (21)$$

If we know $p|_{\Gamma} = \lambda$, we could compute the (\vec{u}, p) solution of (20) by solving $(N+1)$ Dirichlet problems (22) (23)

$$-\Delta p = \vec{v} \cdot \vec{f} \text{ in } \Omega ; p|_{\Gamma} = \lambda \quad (22)$$

$$\alpha u_i - \Delta u_i = \frac{\partial p}{\partial x_i} + f_i \text{ in } \Omega ; u_i|_{\Gamma} = z_i \quad i = 1, N \quad (23)$$

Or the ϕ solution of (24)

$$\Delta \phi = \vec{v} \cdot \vec{u} \text{ in } \Omega ; \phi|_{\Gamma} = 0 \quad (24)$$

By taking the Laplacian Δ of (24) and by using (22) (24), we have

$$\Delta \Delta \phi = \Delta (\vec{v} \cdot \vec{u}) = \vec{v} \cdot (\Delta \vec{u}) = -\Delta p - \vec{v} \cdot \vec{f} - \alpha \vec{v} \cdot \vec{u} = -\alpha \Delta \phi \quad (25)$$

Or otherwise

$$\Delta \Delta \phi + \alpha \Delta \phi = 0 ; \phi|_{\Gamma} = 0 \quad (26)$$

Setting λ so that $\frac{\partial \phi}{\partial n}|_{\Gamma} = 0$

then $\phi \equiv 0$ & $\vec{v} \cdot \vec{u} = 0$

Applying $\lambda \rightarrow \frac{\partial \phi}{\partial n}|_{\Gamma}$ defined by refining (21)-(24) we have A and b such that

$$\frac{\partial \lambda}{\partial n} \Big|_{\Gamma} = A\lambda + b \quad (27)$$

We must thus select λ defined on the boundary as the solution to the linear problem (28)

$$A\lambda + \tilde{b} = \tilde{0} \quad (28)$$

Remark: The solution of (28) is immediate, as A is symmetrical and highly elliptical. The functional support relating to these results is expanded in [9].

Construction of A

The linear operator A is implicitly defined by the sequence of Dirichlet problems (29) (30) (31) (32)

$$\Delta p_{\lambda} = 0 \quad \text{in } \Omega ; p_{\lambda} \Big|_{\Gamma} = \lambda \quad (29)$$

$$\alpha \tilde{u}_{\lambda} - \Delta \tilde{u}_{\lambda} = -\tilde{\nabla} p_{\lambda} \quad \text{in } \Omega ; \tilde{u}_{\lambda} \Big|_{\Gamma} = 0 \quad (30)$$

$$-\Delta \phi_{\lambda} = \tilde{\nabla} \cdot \tilde{u}_{\lambda} \quad \text{in } \Omega ; \phi_{\lambda} \Big|_{\Gamma} = 0 \quad (31)$$

$$A\lambda = -\frac{\partial \phi_{\lambda}}{\partial n} \Big|_{\Gamma} \quad (32)$$

Construction of b

b is explicitly defined by the sequence of Dirichlet problems (33) (34) (35)

$$\Delta p_0 = \tilde{\nabla} \cdot \tilde{f} \quad \text{in } \Omega ; p_0 \Big|_{\Gamma} = 0 \quad (33)$$

$$\alpha \tilde{u}_0 - \Delta \tilde{u}_0 = \tilde{f} - \tilde{\nabla} p_0 \quad \text{in } \Omega ; \tilde{u}_0 \Big|_{\Gamma} = \tilde{z} \quad (34)$$

$$-\Delta \phi_0 = \tilde{\nabla} \cdot \tilde{u}_0 \quad \text{in } \Omega ; \phi_0 \Big|_{\Gamma} = 0 \quad (35)$$

$$b = \frac{\partial \phi}{\partial n} \Big|_{\Gamma} \quad (36)$$

the (\vec{u}, p) solution of (20) is given by (37)

$$\vec{u} = \vec{u}_0 + \vec{u}_\lambda ; \quad p = p_0 + p_\lambda \quad (37)$$

with the trace of λ of $p|_{\Gamma}$ as the only solution to the linear variational equation (38)

$$\langle A\lambda, \mu \rangle = \langle b, \mu \rangle \quad \forall \mu \quad \text{defined in } \Gamma \quad (38)$$

IV. APPROXIMATION OF THE NAVIER-STOKES EQUATIONS BY THE FINITE ELEMENTS METHOD /10

If Ω_h designates a polygonal approximation of the domain occupied by the fluid, \mathcal{T}_h is the set of triangles (T_k) or TRIANGULATION, such that in a standard way:

$$\Omega_h = \bigcup_k T_k ; \quad T_i \cap T_j = \emptyset \quad \text{if } i \neq j$$

h being the largest side.

The infinite functional spaces are substituted by the following finite spaces:

$$V_h^1 = \{ \vec{v}_h \in C^0(\bar{\Omega}_h)^N / \vec{v}_h|_T = \text{polynomial with degree } 1 \quad \forall T \in \mathcal{T}_{h/2} \} \quad (39)$$

where $\mathcal{T}_{h/2}$ is the triangulation obtained from \mathcal{T}_h by dividing each triangle into 4 sub-triangles by drawing a line between the middle of the sides

$$V_h^2 = \{ \vec{v}_h \in C^0(\bar{\Omega}_h)^N / \vec{v}_h|_T = \text{polynomial with degree } 2 \quad \forall T \in \mathcal{T}_h \} \quad (40)$$

$$P_h^1 = \{ p_h \in C^0(\bar{\Omega}_h) / p_h|_T = \text{polynomial with degree } 1 \quad \forall T \in \mathcal{T}_h \} \quad (41)$$

$$M_h^1 = \{ \lambda_h \in C^0(\Gamma_h) / \lambda_h|_B = \text{polynomial with degree } 1 \quad \forall B \in \mathcal{T}_h \} \quad (41')$$

It is stated that the approximation of the fluid velocity is linear (or quadratic) on the triangulation $\mathcal{T}_{h/2}$ (resp. \mathcal{T}_h) and that the approximation of the pressure is linear on the triangulation \mathcal{T}_h .

The problem is limited to discrete spaces (V_h^1, P_h^1, M_h^1) with the speed and pressure of independent variables.

If N_2 designates the number of nodes of the triangulation $\mathcal{T}_{h/2}$, N_1 is the number of nodes of the triangulation \mathcal{T}_h and N_{1F} the number of boundary nodes, whereas the number of unknowns of the discrete state system is $N \times N_2 + N_1 + N_{1F}$ with $N = 2, 3$ space dimension.

In the sequence of equations of the Stokes algorithm, the linear solutions $D_h, H_h = d_h$ are of three types:

1) Discrete variational formulation of the state equation in velocity

$$\begin{aligned} & \int_{\Omega_h} \tilde{u}_h \cdot \tilde{v}_h dx + \int_{\Omega_h} \tilde{u}_h \cdot \tilde{\nabla} \tilde{v}_h dx - \int_{\Omega_h} \tilde{p}_h \tilde{\nabla} \tilde{v}_h dx + \int_{\Omega_h} \tilde{f} \cdot \tilde{v}_h dx \quad \forall \tilde{v}_h \in V_h^1 \\ & \tilde{u}_h|_{\Gamma_h} = \tilde{z}_h \end{aligned} \quad (42)$$

write (42) for $\tilde{v}_h = (N_i) \quad i = 1, \underline{N_2}$, using radix V_h^1

2) Discrete variational formulation of a pressure equation

$$\int_{\Omega_h} \tilde{\nabla} \tilde{p}_h \cdot \tilde{\nabla} \tilde{\phi}_h dx = \int_{\Omega_h} \tilde{\nabla} \cdot \tilde{f}_h \tilde{\phi}_h dx \quad ; \quad \tilde{p}_h|_{\Gamma_h} = \lambda_h \quad (43)$$

write (43) for $\tilde{\phi}_h = (M_i) \quad i = 1, \underline{N_1}$, using radix P_h^1

3) Discrete variational formulation of a pressure trace equation

$$\int_{\Gamma_h} \frac{\partial \tilde{\phi}}{\partial n} \lambda_h d\Gamma = \int_{\Gamma_h} \frac{\partial \tilde{\phi}}{\partial n} u_h d\Gamma \quad \forall u_h \in M_h \quad (44)$$

write (44) for $u_h = (w_i) \quad i = 1, \underline{N_{1F}}$, using radix M_h

During an optimal control iteration, the number of discrete Stokes problems is approximately 5 /12

- one Stokes problem for the state equation
- one Stokes problem for the calculation of the gradient of criterion J'
- ~ 3 Stokes problems for the search for the optimal control step \vec{w} and for the direction \vec{h} pre-defined $\min_{\lambda} J(\vec{w} - \lambda \vec{h})$

Each Stokes problem requires $(2 \times N + 4)$ discrete scalar Dirichlet problems as well as the solution of the small system (44). The empty Dirichlet matrices corresponding to $(-\Delta)$ or $(\alpha \text{ Id} - \Delta)$ and the small complete matrices relatively to the pressure trace remain the same during the iterations and are thus factored out, once and for all (by a Cholevski direct type method) outside of the control loop. The construction of the small complete matrix (44) is a preliminary operation costing $N1Fx (N+2)$ Dirichlet problems.

At each time cycle $n\Delta t$, several optimal control iterations (~ 5) are required to ensure the convergence, the initial solution being the result of the preceding cycle $(n-1) \Delta t$.

Numerical Simulations

The optimal control method has been tested numerically on the I.E.M. 370/168. The flow visualizations are the velocity field, the intensity of the rotational, the streamlines and the pressure fields in the case 2-D, the separated zones and the tube lines of the rotational for applications 3-D.

The characteristics of each calculation (nodes, elements, Cholevski /13 coefficient number, Reynolds number, calculation time C.P.U) are shown on the output plots of the code. The triangulations are produced by the MODULEF techniques [10]. The large number of solutions $AX = B$

to be found, with constant A, explains the preference given to a direct 'CHOLEVSKI' or 'SKYLINE' method. The matrice band widths are minimized by the CUTHIL-MACKEE [11] algorithms and A is factored out, once and for all, in a main core for simple test cases saved on secondary storage (disks) for complex industrial configurations.

The important role played by the Stokes solution (or of incompressible idealized fluid) should be brought to light as a forecast for an internal flow estimate (or external) with a small Reynolds number.

First of all, the validity of the code was tested on simple examples such as:

1. The flow 2-D in a conduit with sudden enlargement to verify the convergence toward the steady state of the solution after several time cycles ($Re = 100$) Figures 1, 2, 3. Comparisons with the A. G. HUTTON code [12] are presented.
2. The unsteady 2-D flow around a Reynolds circle 200 with numerical perturbation after several time cycles (≈ 10) to produce behind the circle the Karman path composed of alternating eddies Figure 4.

Finally, the flows on more complex geometries representing industrial configurations and requiring larger calculation times ($> 1h$ CPU) were simulated.

3. The separated flow 2-D around a profile of an air intake with a high incidence is analyzed at various time cycles. Figures 5 through 13 show the origin, the development, the path and the disappearance of the eddies behind the profile (Re relative to the chord), on the internal and upper external part of the air inlet (Re relating to the maximum deviation) and their effect on pressure.

4. The separated flow 3-D behind a sphere and around a sweptback wing with a high incidence. Figures 14 through 17 show the domain of return flow ($u_1 < 0$ with $U = (u_1, u_2, u_3)$) and several views of the tube lines of the rotational originating from the separated zone are presented. /14

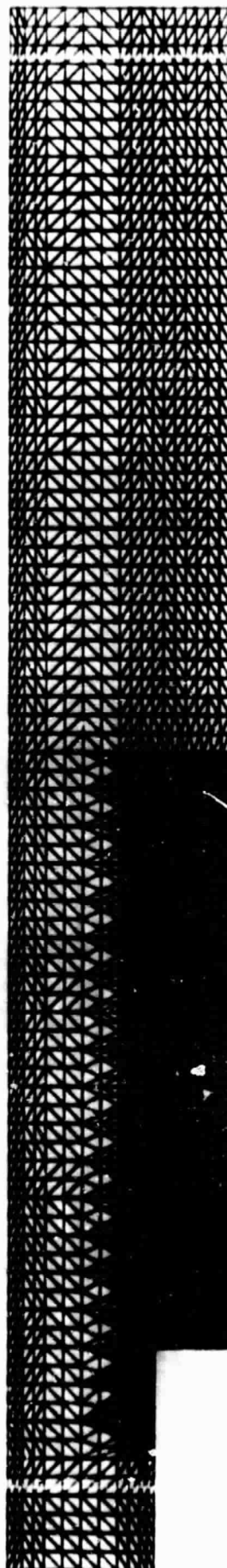
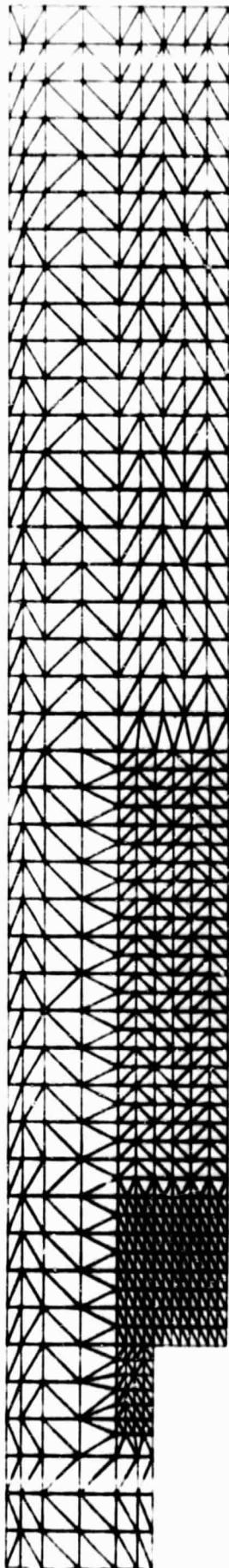
ACKNOWLEDGMENTS

This study, conducted jointly by the LABORIA-IRIA (R. GLOWINSKI, O. PIRONNEAU) and by the AMD-BA industries provides a basis for future research and is partially supported by the D.R.E.T. (Contracts 77/213, 77/410)

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8. IRIA-LABORIA - Utilisation du contrôle optimal pour la résolution numérique des équations de Navier-Stokes. DRET 77/213 Lot 1 Avril 1978 (Application of optimal control for the numerical solution of Navier-Stokes equations. DRET 77/213 April 1978 part 1.
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\mathcal{E}_1 2346 4436 194571

\mathcal{E}_h 619 1109 21634

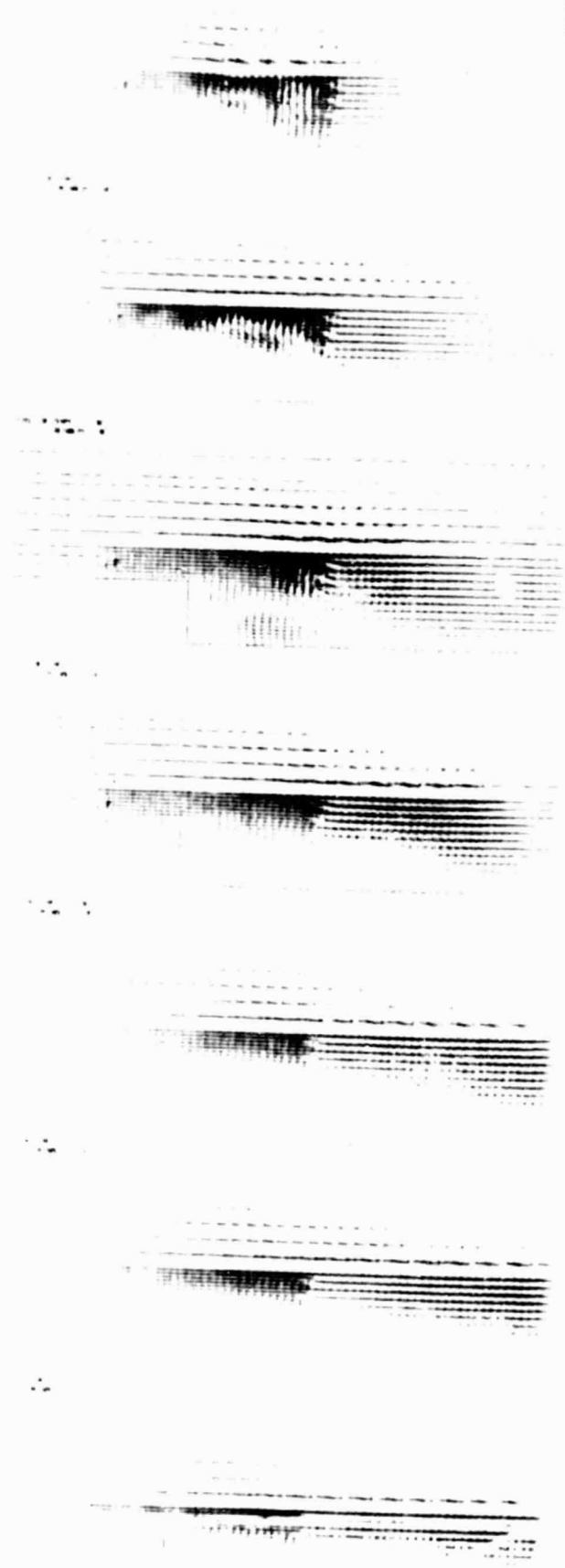
* NODES :
ELEMENTS :
COEF. CHOL :

*NODES

TRIANGULATIONS \mathcal{E}_h et \mathcal{E}_1

Figure 1

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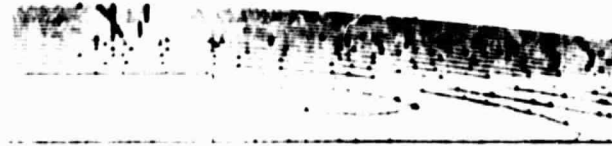
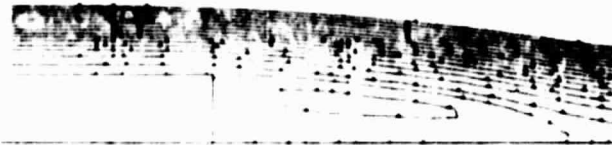
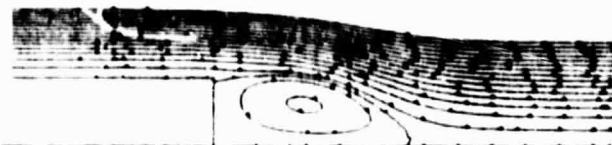
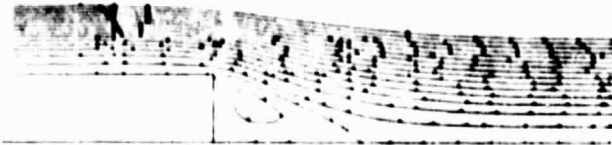
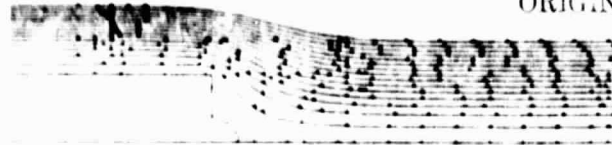
FLOW OVER A STEP
VELOCITY FIELDS

**ÉCOULEMENT SUR UNE MARCHÉ
CHAMPS DES VITESSES**

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Figure 2

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FLOW OVER A STEP
STREAMLINES

ÉCOULEMENT SUR UNE MARCHÉ
LIGNES DE COURANTS

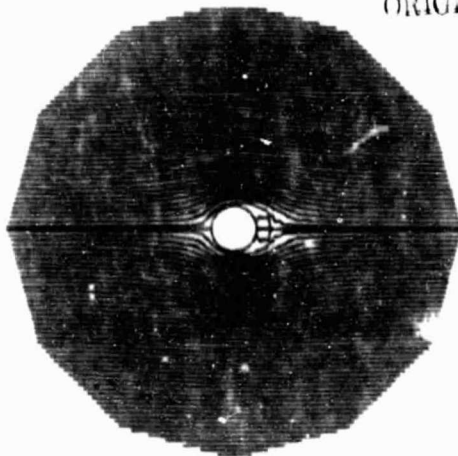
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Figure 3

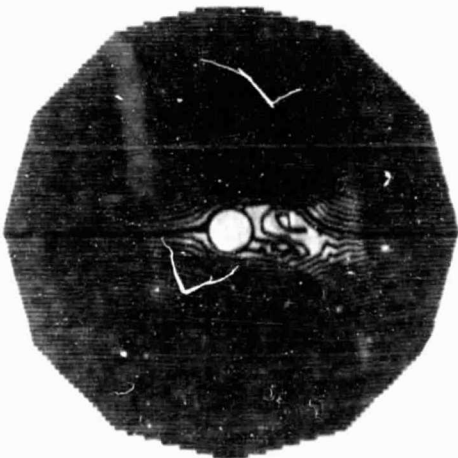
ÉCOULEMENT AUTOUR D'UN CYLINDRE

FLOW AROUND A CYLINDER

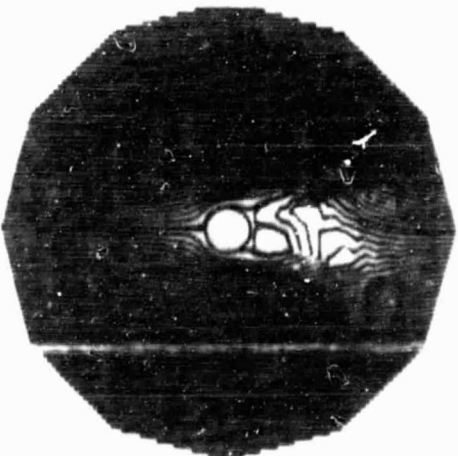
* CYCLE DE TEMPS 10
REYNOLDS 200.



CYCLE DE TEMPS 30
REYNOLDS 200.

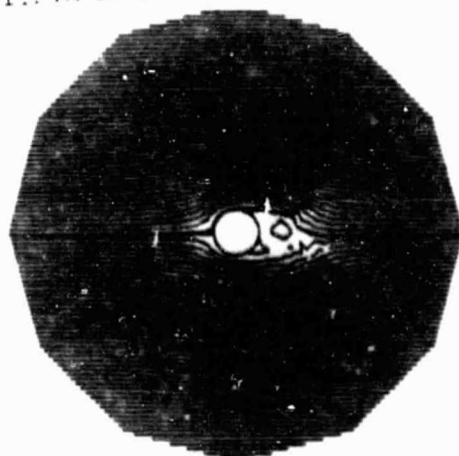


CYCLE DE TEMPS 50
REYNOLDS 200.



CYCLE DE TEMPS 20
REYNOLDS 200.

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CYCLE DE TEMPS 40
REYNOLDS 200.



CYCLE DE TEMPS 60
REYNOLDS 200.

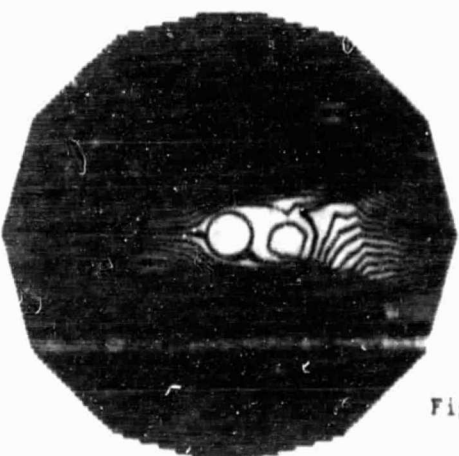


Figure 4

* time cycle

FLOW AROUND A PROFILE
ÉCOULEMENT AUTOUR D'UN PROFIL

$\alpha = 30^\circ$

*CYCLE DE TEMPS 30
REYNOLDS 200.

CYCLE DE TEMPS 40
REYNOLDS 200.

*time cycle



CYCLE DE TEMPS 50
REYNOLDS 200.

CYCLE DE TEMPS 60
REYNOLDS 200.



CYCLE DE TEMPS 70
REYNOLDS 200.

CYCLE DE TEMPS 80
REYNOLDS 200.

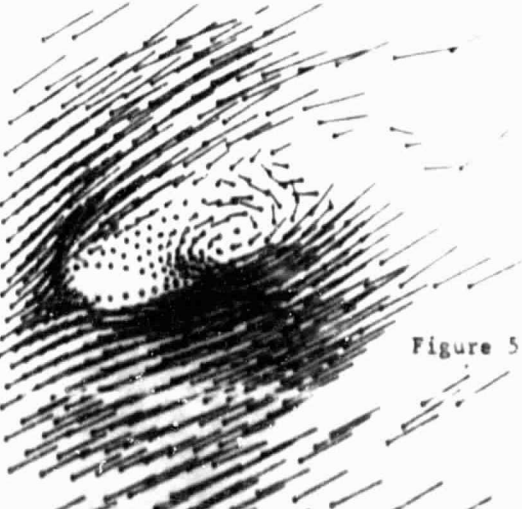


Figure 5

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\mathcal{C}_h TRIANGULATION AROUND AN INLET AT LARGE INCIDENCE

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ELEMENTS : 1458
CHOL. COEF. : 101370

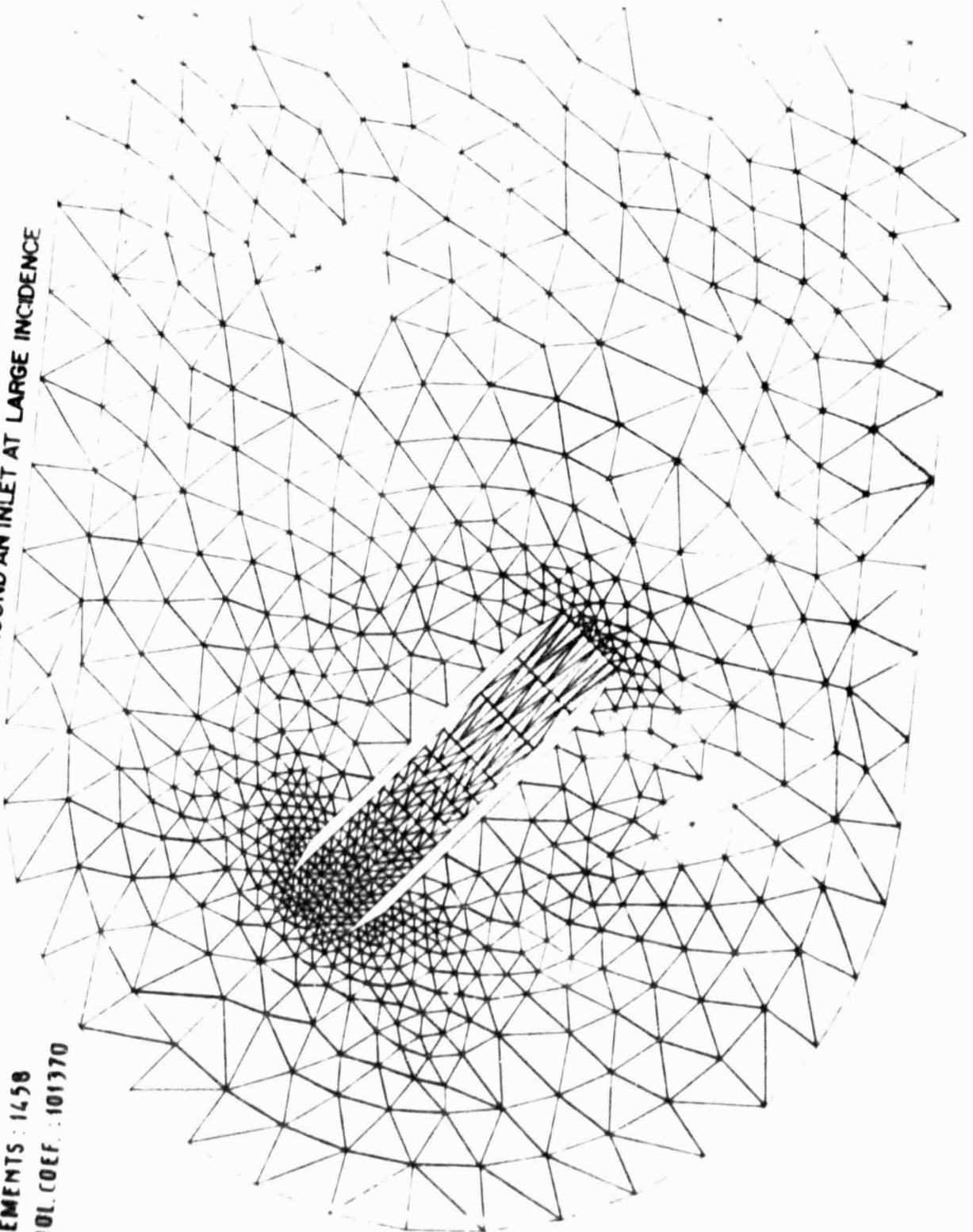
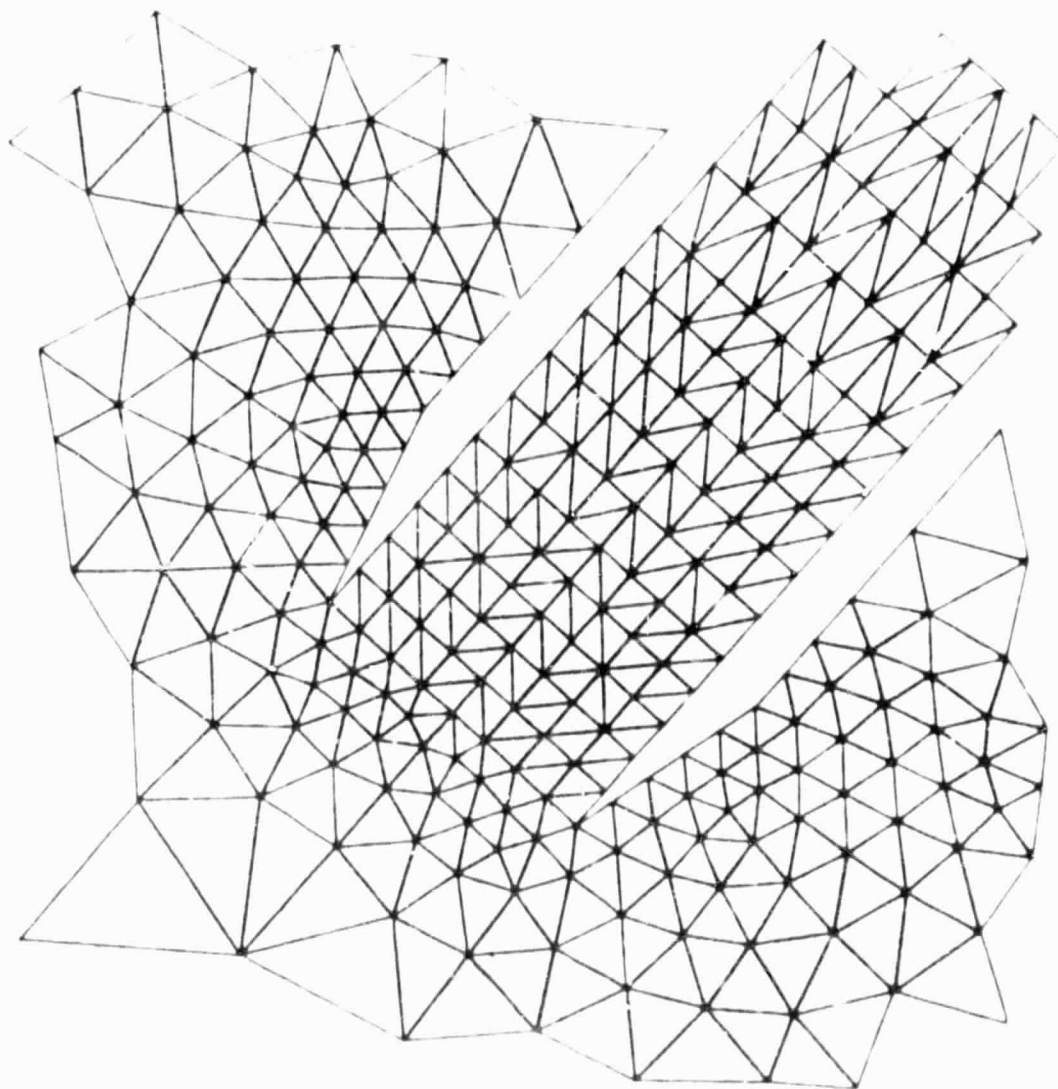


Figure 6

C_h ENLARGEMENT AROUND AN INLET



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ORIGINAL FIGURE IS FOR

Figure 7

REPRODUCTION OF THE
ORIGINAL IMAGE IS POOR

ϕ_h TRIANGULATION AROUND AN INLET AT LARGE INCIDENCE

NODES : 3049
ELEMENTS : 5032
CHOL COEF : 314685

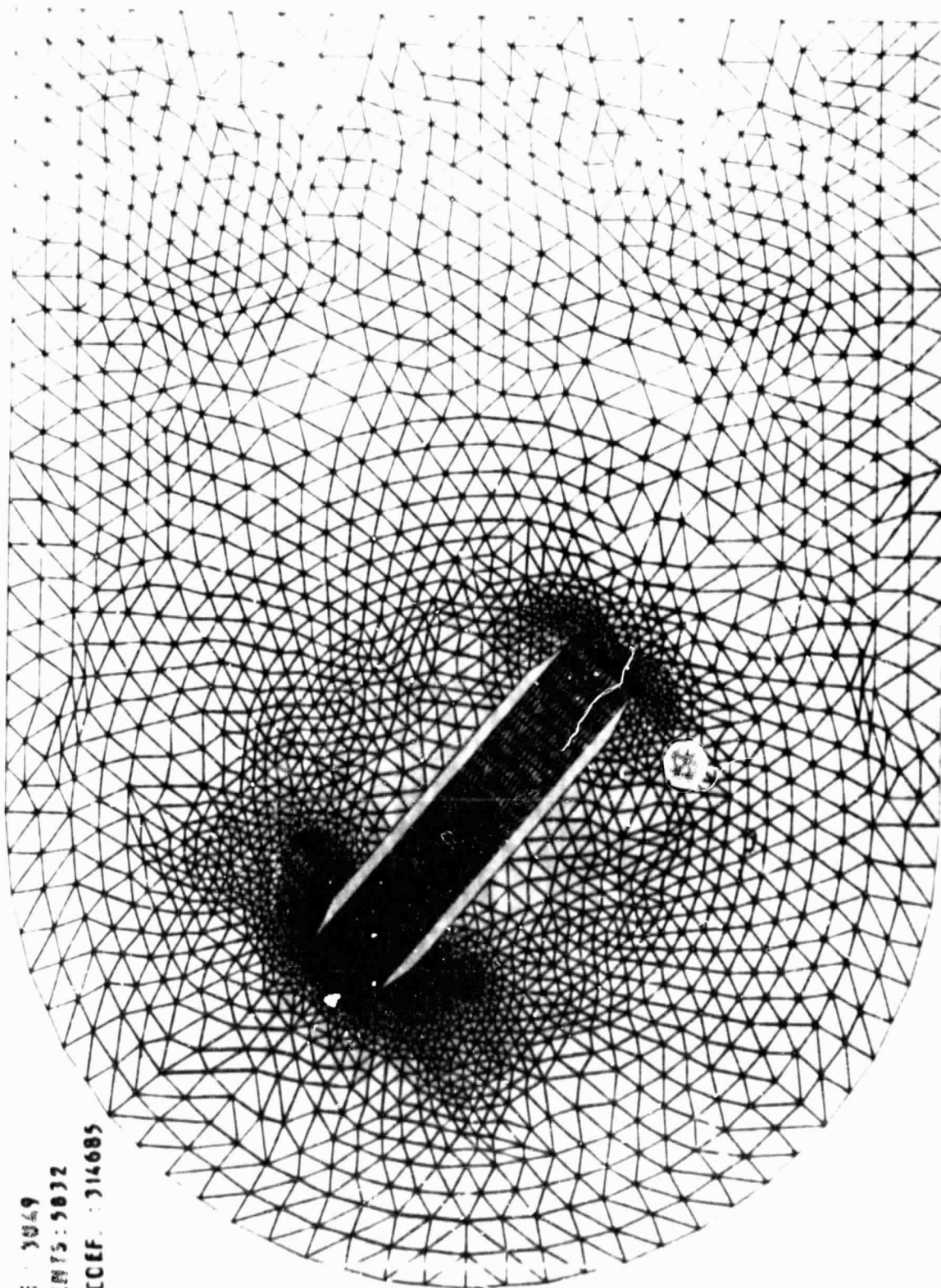
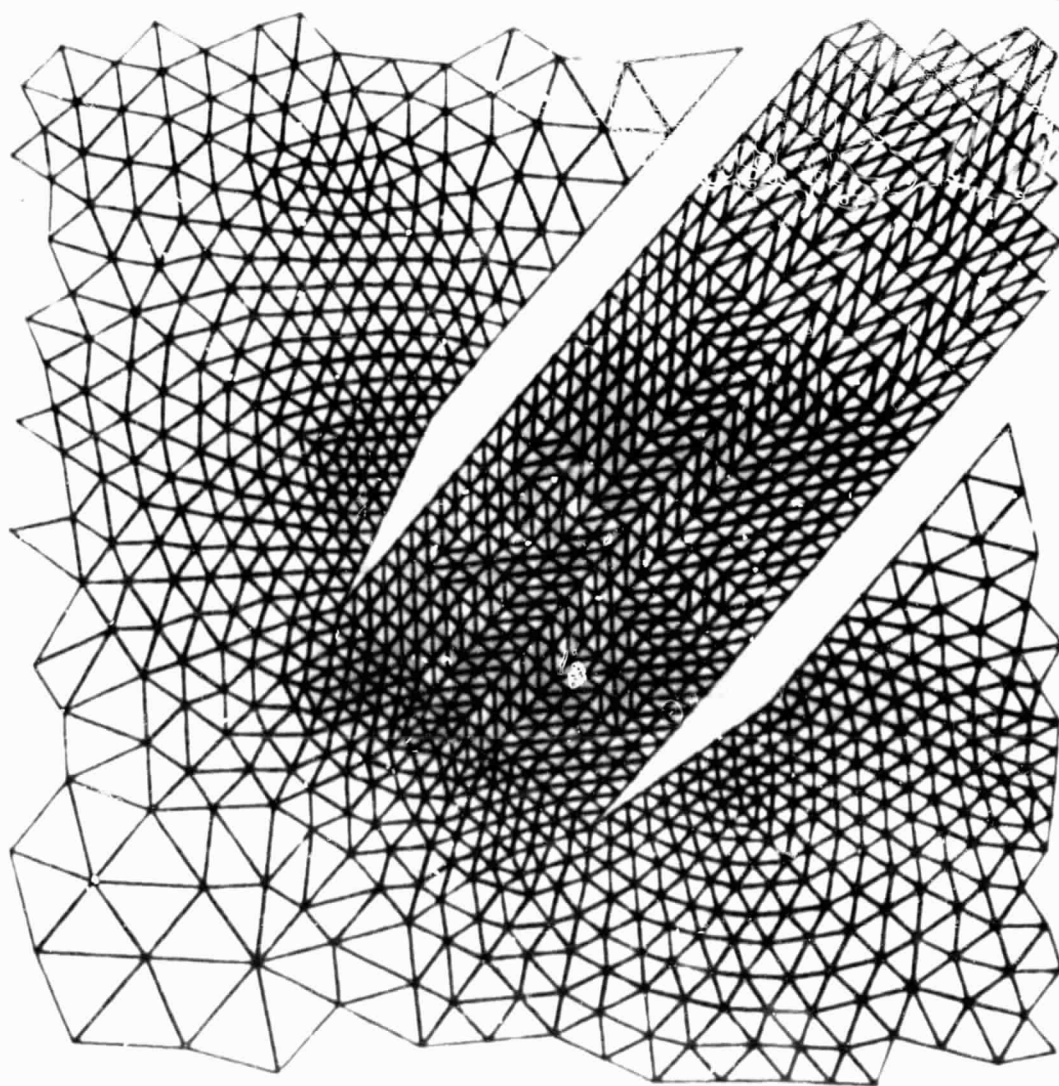


Figure 8

20 C_h ENLARGEMENT AROUND AN INLET



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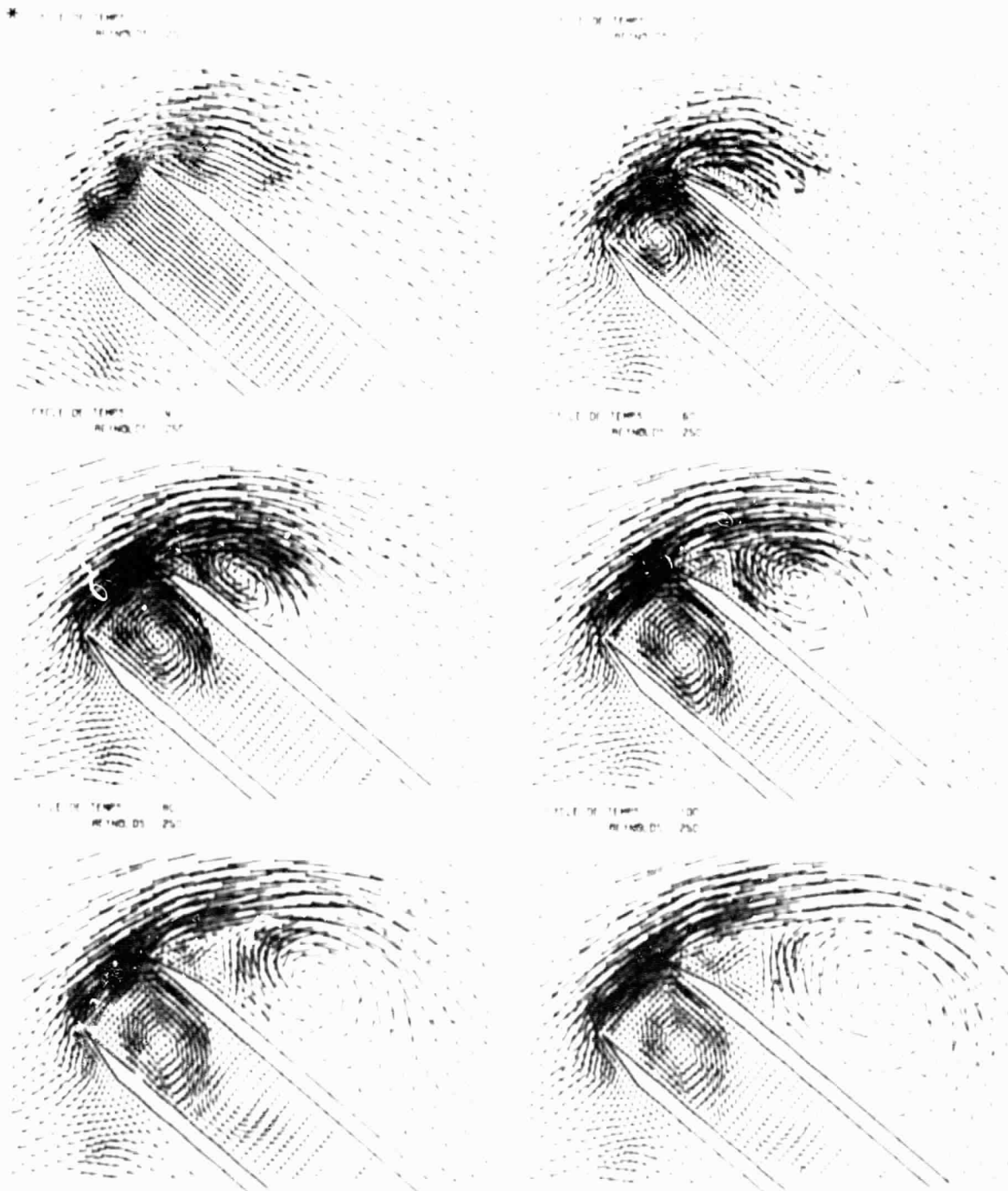
Figure 9

flow ar FLOW AROUND AN AIR INLET

ÉCOULEMENT AUTOUR D'UNE ENTRÉE D'AIR

$\alpha = 40^\circ$ CHAMPS DES VITESSES VELOCITY FIELDS

*time cycle

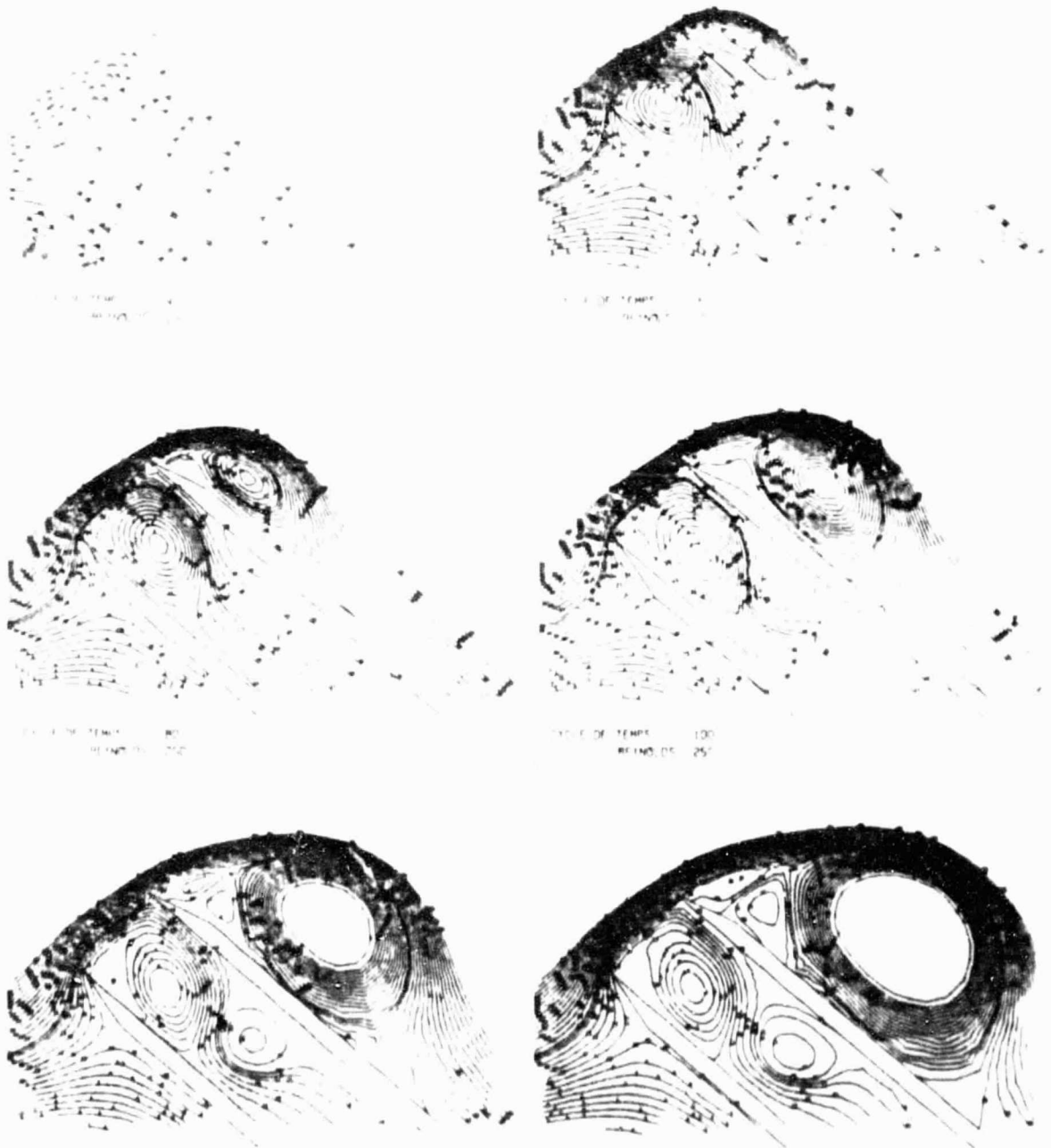


FLOW AROUND AN AIR INLET
 ECOULEMENT AUTOUR D'UNE ENTRÉE D'AIR

$\alpha = 40^\circ$ LIGNES DE COURANT STREAMLINES

*time cycle
 *

REPRODUCTION OF THE
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REPRODUCTION OF THE ORIGINAL IS POOR

Figure 12

FLOW AROUND AN AIR INLET
ÉCOULEMENT AUTOUR D'UNE ENTRÉE D'AIR

$\alpha = 40^\circ$

ISO-PRESSION

PRESSURE

*time cycle

*

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Figure 13

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SEPARATED FLOW AROUND A SPHERE FINITE P1/P2 ELEMENT METHOD WITH OPTIMAL CONTROL THEORY

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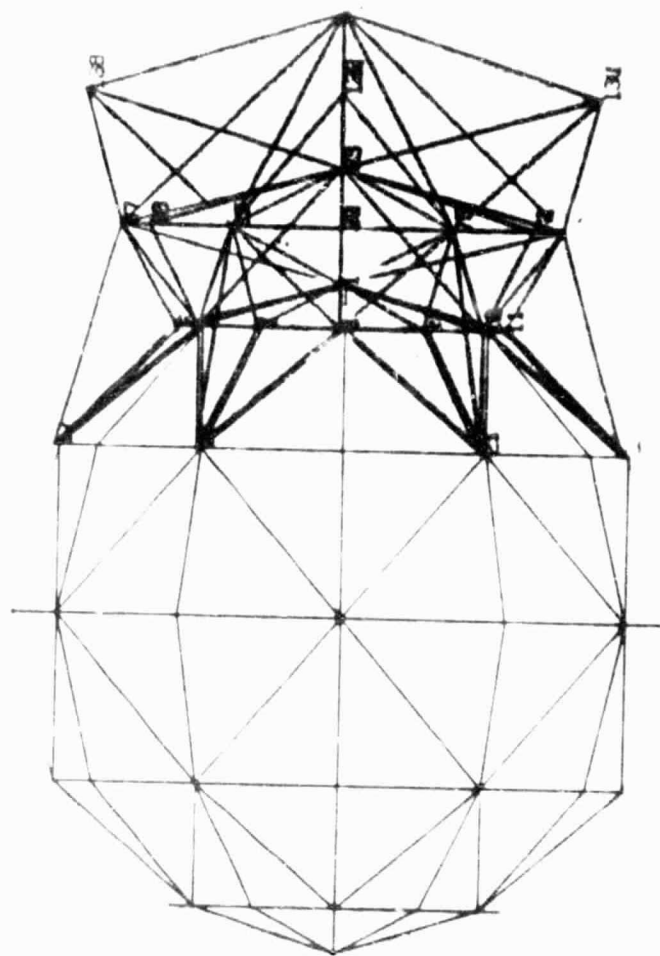


Figure 14

FLOW AROUND AN AIR INLET
 ECOULEMENT AUTOUR D'UNE ENTRÉE D'AIR

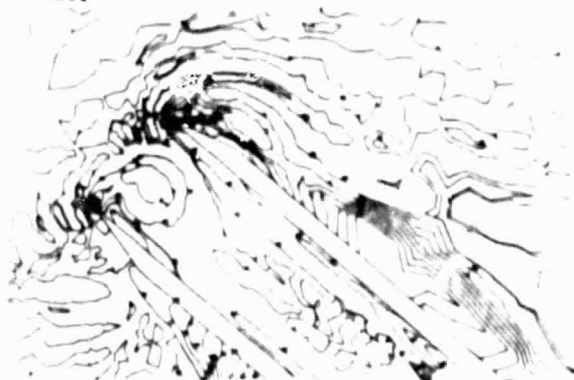
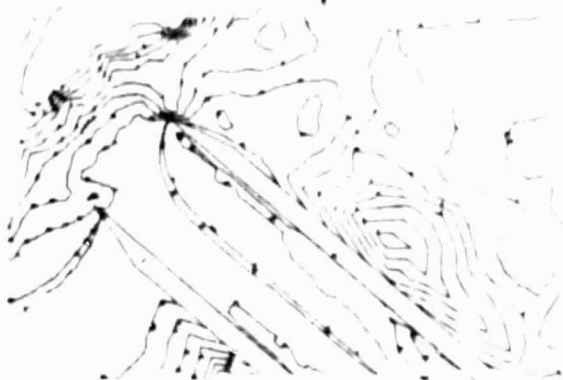
$\alpha = 40^\circ$ ISO-TOURBILLON EDDY

*time cycle

* CYCLE DE TEMPS
 RETARD DS 250

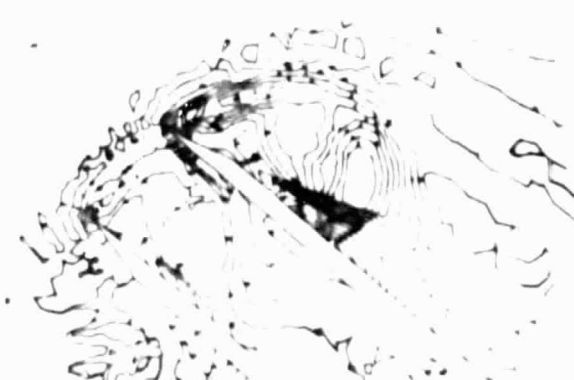
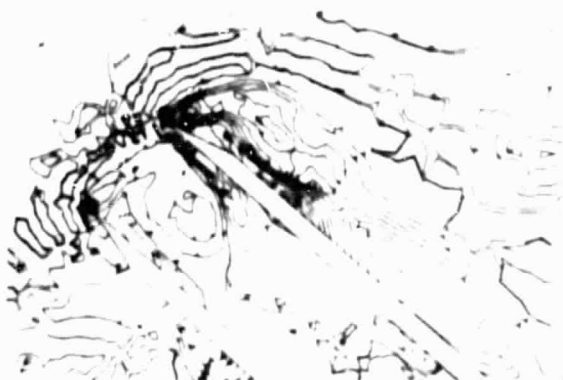
REPRODUCIBILITY OF THE
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CYCLE DE TEMPS 20
 RETARD DS 250



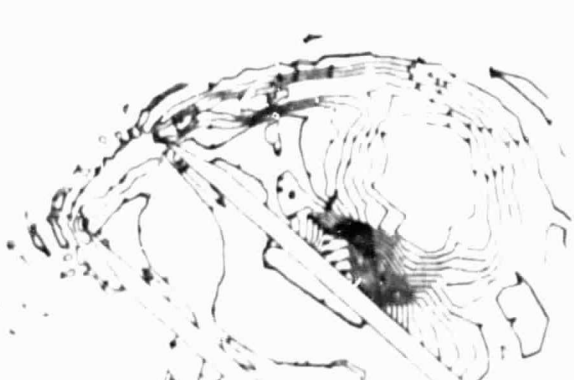
CYCLE DE TEMPS 40
 RETARD DS 250

CYCLE DE TEMPS 60
 RETARD DS 250



CYCLE DE TEMPS 80
 RETARD DS 250

CYCLE DE TEMPS 100
 RETARD DS 250



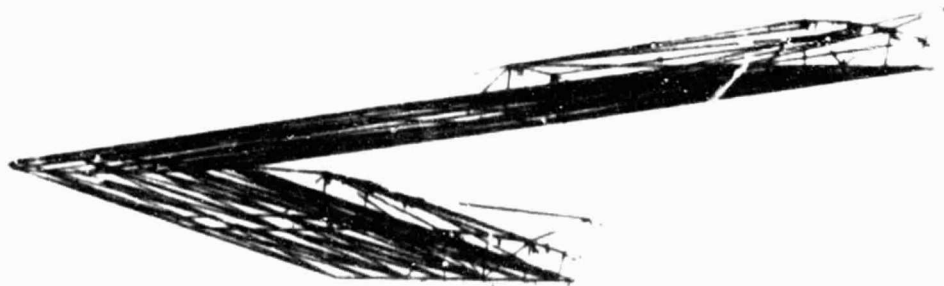
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Figure 11

3^D NUMERICAL SIMULATION OF VISCOUS SEPARATED FLOW AROUND A IDEALIZED WING

Reynolds 200 - Incidence 30°

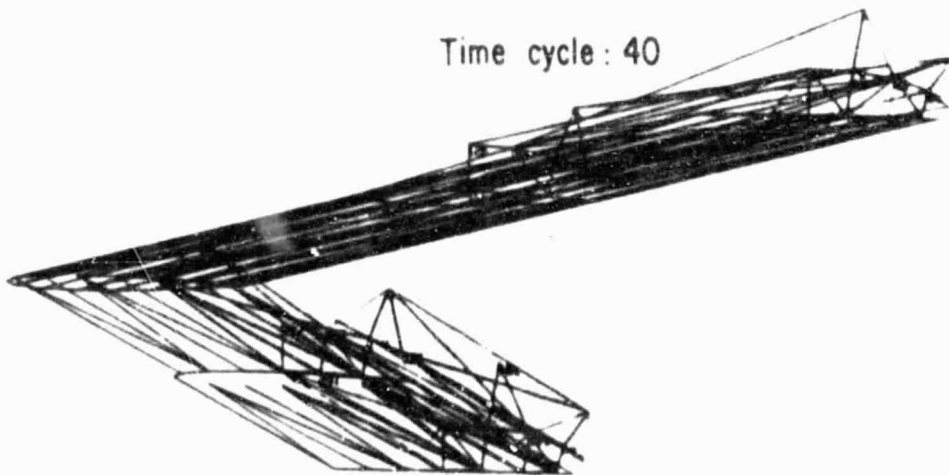
*The triangles show the domain where the u-component of the
velocity is negative*



Time cycle : 20



Time cycle : 40



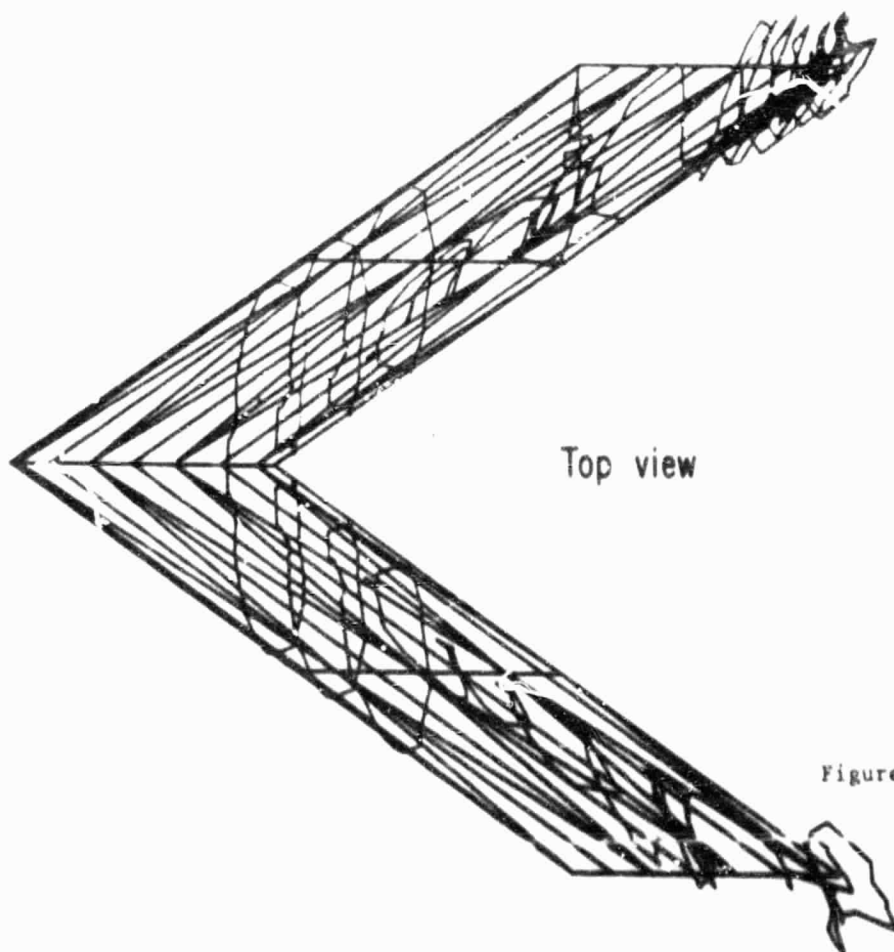
DIFFERENT VIEWS OF VORTICITY LINES IN THE REVERSE DOMAIN



Side view



Rear view



Top view

Figure 16